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Effective action for phase fluctuations in d-wave superconductors near a Mott transition

D. Bensimon and R. Zeyher[a]

Max-Planck-Institut für Festkörperforschung, Heisenbergstrasse 1, 70569 Stuttgart, Germany

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Abstract. Phase fluctuations of a d-wave superconducting order parameter are theoretically studied in the context of high-T_c cuprates. We consider an extended $t-J$ model describing electrons in a layer which also contains long-range Coulomb interactions. The constraint of having at most singly occupied sites is enforced by an additional Hubbard term. The Heisenberg interaction is decoupled by a d-wave order parameter in the particle-particle channel. Assuming first that the equilibrium state has long-range phase order, the effective action $\mathcal{S}_{eff}$ is derived perturbatively for small fluctuations within a path integral formalism, in the presence of the Coulomb and Hubbard interaction terms. In a second step, a more general derivation of $\mathcal{S}_{eff}$ is performed in terms of a gradient expansion which only assumes that the gradients of the order parameter are small whereas the value of the phase may be large. We show that in the phase-only approximation the resulting $\mathcal{S}_{eff}$ reduces in leading order in the field gradients to the perturbative one which thus allows to treat also the case without long-range phase order or vortices. Our result generalizes previous expressions for $\mathcal{S}_{eff}$ to the case of interacting electrons, is explicitly gauge invariant, and avoids problematic singular gauge transformations.

PACS. 74.40.+k Fluctuations (noise, chaos, nonequilibrium superconductivity, localization, etc.) – 74.72.-h Cuprate superconductors (high-T_c and insulating parent compounds) – 74.25.Jb Electronic structure – 74.25.Dw Superconductivity phase diagrams

1 Introduction

The nature of the pseudogap in underdoped high-T_c oxides and its relation to superconductivity remains one of the unsolved problems in high-T_c superconductivity. One scenario assumes that the physics in the underdoped and optimally doped region is mainly determined by the competition of the superconducting order parameter with a second one in the particle-hole channel. Possible candidates are antiferromagnetism, charge- and spin-density waves [1,2], in particular with an internal d-wave symmetry [3,4], and stripes [5]. One experimental constraint is that this second order parameter is strongly anisotropic, being large along the k_x and k_y axis of the Brillouin zone, and practically zero near the diagonal $k_x = k_y$. This requirement is most naturally fulfilled if this order parameter has d-wave symmetry like the superconducting order parameter. A second scenario assumes that no instabilities or strong fluctuations in the particle-hole channel are relevant in the underdoped regime but that phase (and perhaps amplitude) fluctuations of the superconducting order parameter are important in this region [6].

Presently, it is unclear which of the above two scenarios is more realistic in describing high-T_c cuprates. In this paper we want to explore some aspects of the second approach in more detail. While it seems that the first scenario has become more popular than the second one we think that there are enough reasons to study phase fluctions of the superconducting order parameter. For instance, order parameters due to structural phase transitions with finite momentum transfers may easily be anisotropic because of strongly varying nesting properties along the Fermi surface. However, whether one can achieve along this route the observed large anisotropy or even d-wave symmetry of the pseudogap in a generic way remains unclear. Experimental SIN tunneling data in the cuprates also show a rather symmetric one-particle density of states, both in the superconducting and the pseudogap region, with respect to the chemical potential [7,8]. Such a behavior is characteristic for superconductivity but not generic for densities with structural order parameters in the particle-hole channel, especially, if they are associated with large momentum transfers. These problems vanish if one assumes that only one complex order parameter related to the observed d-wave superconductivity is relevant.

Fluctuations effects in superconductors are often described by the time-dependent Ginzburg-Landau equations [9]. They describe a relaxational behavior of the order parameter back to equilibrium, i.e., their time-dependent part is of first order in the time derivative and independent of momentum. In the pure system they are most successfull near T_c and at low frequencies compared

[a] e-mail: r.zeyher@fkf.mpg.de

to the temperature. In the following we want to consider a different regime, namely, temperatures substantially below the mean-field transition to superconductivity where the mean-field order parameter is large but varies slowly in space and time because of a small phase stiffness constant Λ [10]. The t-J model and its constraint to have no doubly occupied sites leads necessarily to small values for Λ's at small dopings [11]. At low temperature Λ is determined by the diamagnetic term which is given by the Fourier transform of the one-particle momentum distribution function at non-vanishing lattice vectors. The constraint of having no doubly occupied sites implies that the maximum occupation of a momentum state is $1+\delta$ instead of 2 as in the free case [12]. This together with the sum rule shows that the diamagnetic term must vanish linearly in δ in the limit $\delta \to 0$. As a result large phase fluctuations should occur at small dopings. In agreement with this picture we derive in the present work an effective action for phase fluctuations not as a power but as a gradient expansion in the phase. The method we follow also allows to take into account interaction terms between the electrons such as the Hubbard or the long-range Coulomb interaction. This feature is important because the small phase stiffness at low dopings is caused by correlation effects. At the same time our treatment leads to an explicitly gauge invariant form for the effective action for phase fluctuations.

Most of the previous derivations of the effective action for phase fluctuations employed singular gauge transformations [13–18]. Such a transformation means in the static case for the Bogoliubov equations that $\Delta(\boldsymbol{r}) \to \Delta(\boldsymbol{r})\cdot e^{-\mathrm{i}\theta(\boldsymbol{r})}$, $u(\boldsymbol{r}) \to u(\boldsymbol{r})\cdot e^{-\mathrm{i}\theta(\boldsymbol{r})/2}$, $v(\boldsymbol{r}) \to v(\boldsymbol{r})\cdot e^{\mathrm{i}\theta(\boldsymbol{r})/2}$, $\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(\boldsymbol{r}) - (\hbar c/2\mathrm{i}e)\nabla\theta(\boldsymbol{r})$ [19]. Δ, u, v are the pair, electron and hole wave functions, respectively, $\mathbf{A}$ the vector potential, and θ is an arbitrary real function. This transformation allows to remove completely the fluctuations in Δ and to describe them as fluctuations in $\mathbf{A}$. However, Δ, u and v have to be unique functions of $\boldsymbol{r}$, i.e., their phase can only change by multiples of 2π after passing through a closed loop. For instance, in the case of one vortex the angular dependence of the transformed wave function becomes a linear combination of functions $e^{i\mu\theta}$, where 2μ is an odd and not, as in the free case, an even integer. The change of boundary conditions associated with singular gauge transformations affects thus wave and Green's functions even far away from the center of the vortex in a complicated manner. On the other hand the importance of these effects for physical quantities [20] is presently not clear, most authors neglect this problem whereas, according to references [21–23], it causes very interesting effects such as non-Fermi liquid behavior, power laws of correlation functions, etc., in the normal state. In order not to violate the basic requirement of uniqueness of wave functions we avoid completely problematic singular gauge transformations and derive the effective action for phase fluctuations by means of a gradient expansion in the order parameter.

The paper is organized as follows. In Section 2 the Hamiltonian is specified and phase fluctuations of the order parameter are introduced. In Section 3 the effective action for phase fluctuations is derived under the assumption that the phases θ deviate only little from a homogenous state, for instance, $\theta = 0$. The microscopic quantities appearing in the action are one- and two-particle Green's functions associated with the Hamiltonian without external potentials or phase fluctuations. A more general derivation for the effective action is given in Section 4 using a gradient expansion which only assumes that the gradients of the order parameter are small whereas θ may be large. It is shown that the two effective actions derived in Sections 3 and 4 are equivalent in the phase-only approximation for the order parameter. Section 5 contains a discussion of the results and the conclusions. In Section 3 relations between different correlation functions describing density, current and pair fluctuations have been inferred from non-singular gauge transformations and used in obtaining the final form for the effective action. As an example we verify in the appendix explicitly one of these relations in the non-interacting case without referring to gauge invariance arguments.

2 Hamiltonian and phase fluctuations of the order parameter

We consider a generalized t-J model [24,25], which also contains repulsive Coulomb and Hubbard interactions, on a Bravais lattice consisting of layers of squares along the x and y axis. Its Hamiltonian is given by

$$\mathcal{H} = -\sum_{i,j,\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} + J \sum_{\langle i,j\rangle} \left\{ \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right\} + \mathcal{H}_{int}, \tag{1}$$

$$\mathcal{H}_{int} = \frac{1}{2}\sum_{i,j} V_{ij} n_i n_j + \frac{U}{2}\sum_i n_i n_i. \tag{2}$$

σ is the SU(2) spin color, $c^{\dagger}_{i\sigma}$ ($c_{i\sigma}$) the creation (annihilation) operator of a spin σ electron on the site i, J the Heisenberg interaction, V_{ij} the Coulomb interaction between the sites i and j, and U a repulsive Hubbard term. t_{ij} is the electronic hopping term between the sites i and j, $\langle\ \rangle$ denotes a pair index for nearest neighbor sites on the same layer. $\mathbf{S}_i$ and n_i are the spin and occupancy number operators of site i, respectively. In order to simplify the notation later we will put $a = \hbar = c = 1$, where a is the lattice constant of the square lattice and c the velocity of light. We also assume that the lattice contains N_s sites. Let us introduce the singlet pair operators

$$B^{\dagger}_{ij} = c^{\dagger}_{i\uparrow} c^{\dagger}_{j\downarrow} - c^{\dagger}_{i\downarrow} c^{\dagger}_{j\uparrow},\ B_{ij} = c_{j\downarrow} c_{i\uparrow} - c_{j\uparrow} c_{i\downarrow}, \tag{3}$$

which respectively create and annihilate a singlet on the bond $\langle i,j\rangle$. $\mathcal{H}$ can then be expressed equivalently as [14]

$$\mathcal{H} = -\sum_{i,j,\sigma} t_{ij} c^{\dagger}_{i\sigma} c_{j\sigma} - \frac{J}{2}\sum_{\langle i,j\rangle} B^{\dagger}_{ij} B_{ij} + \mathcal{H}_{int}. \tag{4}$$

To derive the partition function, we work in a path integral formalism [26–29]. The Heisenberg interaction term is decoupled using the Hubbard-Stratonovitch transformation [30,31], the resulting complex fields Δ^*, Δ correspond to the superconducting order parameter [32]. The partition function of the $t-J$ model (4) in the imaginary time formalism is given by

$$\mathcal{Z} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}\Delta\mathcal{D}\Delta^* \, \exp\bigl(-\mathcal{S}\bigr), \tag{5}$$

with the action

$$\begin{aligned}\mathcal{S} = \int_0^\beta d\tau \Bigg[& \sum_{i,\sigma} \bar{\Psi}_{i\sigma}(\tau)\bigl\{\partial_\tau - \mu\bigr\}\Psi_{i\sigma}(\tau) \\ & - \sum_{i,j,\sigma} t_{ij}\bar{\Psi}_{i\sigma}(\tau)\Psi_{j\sigma}(\tau) \\ & - \sum_{\langle i,j\rangle} \Bigg\{ \frac{1}{4}\bigl[\Delta_{ij}(\tau)\bar{B}^*_{ij}(\tau) + \Delta^*_{ij}(\tau)\bar{B}_{ij}(\tau)\bigr] \\ & - \frac{1}{8J} \mid \Delta_{ij}(\tau) \mid^2 \Bigg\} \Bigg] + \mathcal{S}_{int}. \end{aligned} \tag{6}$$

$\mathcal{S}_{int}$ is the contribution to the action due to $\mathcal{H}_{int}$, β is the thermal factor, $\beta = 1/k_{\mathrm{B}}T$, μ is the chemical potential controlling the electron density while $\bar{\Psi}$, Ψ and $\bar{B}^*$, $\bar{B}$ are Grassmann variables corresponding to the electronic operators $c^\dagger$, c and singlet pair operators $B^\dagger$, B, respectively.

The complex bond variables $\Delta_{i,i+\hat{x}}$ and $\Delta_{i,i+\hat{y}}$ may be written without loss of generality as

$$\begin{aligned} \Delta_{ij}(\tau) &= \mid \Delta_{ij}(\tau) \mid \gamma_{ij}\mathrm{e}^{\mathrm{i}\phi_{ij}(\tau)}, \\ \gamma_{ij} &= \begin{cases} +1 \text{ for } j = i+\hat{x} \\ -1 \text{ for } j = i+\hat{y} \end{cases}, \end{aligned} \tag{7}$$

where $\hat{x}$ and $\hat{y}$ are basis vectors of the direct lattice. The equilibrium d-wave order parameter value is given by

$$\mid \Delta_{ij}(\tau) \mid \equiv \Delta_0, \;\; \phi_{ij}(\tau) \equiv 0, \tag{8}$$

where Δ_0 is a real number independent of i and τ. In the present work, fluctuations of the amplitude are neglected. This implies that only vortices with zero core extension can be discussed after this approximation. Taking appropriate linear combinations of the remaining two phase degrees of freedom per elementary cell yields two different kinds of phase excitations. The first one is in the long-wavelength limit gapless and preserves the local d-wave symmetry whereas the second one is gapped and locally of s-wave symmetry describing phase fluctuations mainly inside the elementary cell. Only the first kind of phase fluctuations is relevant for the long-wavelength, low-frequency limit of the action. It can be described by a real field θ_i which is defined on the original lattice points and related to Φ_{ij} by [14]

$$\phi_{ij}(\tau) = \frac{1}{2}\bigl[\theta_i(\tau) + \theta_j(\tau)\bigr]. \tag{9}$$

A global shift in the the phases θ_i implies a global shift in Φ_{ij}, i.e., the field θ_i is gapless in the long-wavelength limit. Moreover, all field distributions $\{\Phi_{ij}\}$ which can be represented by a continuous envelope function can be well represented by θ_i showing that the field θ_i is a suitable choice in deriving the effective action. The fluctuating order parameter is thus assumed to be

$$\begin{aligned} \Delta_{i,i+\hat{x}}(\tau) &= \Delta_0 \cdot \mathrm{e}^{\mathrm{i}[\theta_i(\tau)+\theta_{i+\hat{x}}(\tau)]/2}, \\ \Delta_{i,i+\hat{y}}(\tau) &= -\Delta_0 \cdot \mathrm{e}^{\mathrm{i}[\theta_i(\tau)+\theta_{i+\hat{y}}(\tau)]/2}. \end{aligned} \tag{10}$$

It is well-known that phase fluctuations induce charge and current fluctuations and thus electric and magnetic fields [33]. Therefore the resulting electromagnetic field has to be considered in a general formalism by including scalar and vector potentials in the action, especially, to guarantee gauge invariance. In the case of lattice models, a minimum coupling scheme to describe the interactions involving the electromagnetic field is given by the Peierls substitution [34], which corresponds to

$$\begin{aligned} -\mu \cdot \bar{\Psi}_{i\sigma}(\tau)\Psi_{i\sigma}(\tau) \longrightarrow & \Bigl[-\mu - e \cdot A_0(\boldsymbol{r}_i,\tau)\Bigr] \\ & \times \bar{\Psi}_{i\sigma}(\tau)\Psi_{i\sigma}(\tau), \end{aligned} \tag{11}$$

$$\begin{aligned} t_{ij} \cdot \bar{\Psi}_{i\sigma}(\tau)\Psi_{j\sigma}(\tau) \longrightarrow & \, t_{ij} \cdot \exp\Biggl(-\mathrm{i}e \int_{\boldsymbol{r}_j}^{\boldsymbol{r}_i} \boldsymbol{A}(\boldsymbol{r},\tau)\cdot d\boldsymbol{l}\Biggr) \\ & \times \bar{\Psi}_{i\sigma}(\tau)\Psi_{j\sigma}(\tau), \end{aligned} \tag{12}$$

where A_0 and $\boldsymbol{A} \equiv (A_x, A_y, A_z)$ are the scalar and vector potentials, respectively, and $(-e)$ is the electron charge. In the case of slowly varying phase fluctuations the induced electromagnetic potentials also vary slowly in time and space so that we can write [35]

$$\int_{\boldsymbol{r}_j}^{\boldsymbol{r}_i} \boldsymbol{A}(\boldsymbol{r},\tau)\cdot d\boldsymbol{l} \approx (\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot\frac{1}{2}\Bigl\{\boldsymbol{A}(\boldsymbol{r}_j,\tau) + \boldsymbol{A}(\boldsymbol{r}_i,\tau)\Bigr\}.$$

3 Effective action for small phase fluctuations

In this section we first give explicit expressions for the change of the action away from the d-wave saddle point due to slow spatial and temporal phase fluctuations and the corresponding induced electromagnetic field. The effective action $\mathcal{S}_{\mathit{eff}}$ is defined by

$$\frac{\mathcal{Z}[A,\theta]}{\mathcal{Z}^{(0)}} = \exp\Bigl(-\mathcal{S}_{\mathit{eff}}[A,\theta]\Bigr), \tag{13}$$

using the abbreviation $A = (A_0, \boldsymbol{A})$. $\mathcal{Z}[A,\theta]$ is the total partition function, containing also the effects due to fluctuations in A and θ. $\mathcal{Z}^{(0)}$ denotes the d-wave saddle point partition function in the presence of the Coulomb and Hubbard terms, and $\mathcal{S}^{(0)}$ its corresponding

action

$$\mathcal{Z}^{(0)} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\Big(- \mathcal{S}^{(0)}\Big), \tag{14}$$

$$\mathcal{S}^{(0)} = \int_0^\beta d\tau \Bigg[\sum_{i,\sigma} \bar{\Psi}_{i\sigma}(\tau)\big\{\partial_\tau - \mu\big\}\Psi_{i\sigma}(\tau) - \sum_{i,j,\sigma} t_{ij}\bar{\Psi}_{i\sigma}(\tau)\Psi_{j\sigma}(\tau) - \sum_{\langle i,j\rangle}\Big\{\frac{1}{4}\Delta_0\gamma_{ij}\big[\bar{B}^*_{ij}(\tau)+\bar{B}_{ij}(\tau)\big] - \frac{1}{8J}\big(\Delta_0\big)^2\Big\}\Bigg] + \mathcal{S}_{int}. \tag{15}$$

By expanding perturbatively the action (6) with respect to the phase exponential factors and electromagnetic potentials appearing in equations (10) and (11), (12), respectively, one obtains the partition function and action related to fluctuations in the phase and the vector potential. We go over to imaginary frequencies and momentum space. The fermionic Matsubara frequencies are labelled by $\mathrm{i}\nu_m$, and the bosonic ones by $\mathrm{i}\omega_n$. It is convenient to define the four-dimensional wave vectors $k = (k_\alpha)_{0\le\alpha\le 3} = (\mathrm{i}\nu_m, \boldsymbol{k})$ and $q = (q_\alpha)_{0\le\alpha\le 3} = (\mathrm{i}\omega_n, \boldsymbol{q})$. The Fourier transformed field variables are defined by

$$\bar{\Psi}_\sigma(\mathrm{i}\nu_m, \boldsymbol{k}) \equiv \bar{\Psi}_\sigma(k) = \frac{1}{\sqrt{\beta N_s}}\sum_i \int_0^\beta d\tau \mathrm{e}^{-\mathrm{i}\nu_m\tau + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i}\bar{\Psi}_{i\sigma}(\tau). \tag{16}$$

It is also convenient to introduce Nambu spinor field operators [36,37] by

$$\bar{\Phi}(k) = \begin{pmatrix}\bar{\Psi}_\uparrow(k) & \Psi_\downarrow(-k)\end{pmatrix}, \qquad \Phi(k) = \begin{pmatrix}\Psi_\uparrow(k) \\ \bar{\Psi}_\downarrow(-k)\end{pmatrix}. \tag{17}$$

In this section we consider only terms up to the second order in the fluctuations θ. This means that we allow only small phase fluctuations around a homogenous state characterized by $\theta = 0$. The partition function and the action, expanded up to second order in A and θ, becomes then

$$\mathcal{Z}[A,\theta] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\Big(- \mathcal{S}[A,\theta]\Big), \tag{18}$$

$$\mathcal{S}[A,\theta] = \mathcal{S}^{(0)} + \mathcal{S}'_A + \mathcal{S}'_\theta, \tag{19}$$

$$\mathcal{S}'_A = \frac{1}{\sqrt{\beta N_s}}\sum_{k,q}\sum_{\alpha=0}^{3} v_\alpha(\boldsymbol{k})A_\alpha(q)\tilde{\Phi}(n_\alpha,k,q) + \frac{1}{\beta N_s}\sum_{k,q,q'}\sum_{\alpha,\alpha'=1}^{3} m^{-1}_{\alpha\alpha'}(\boldsymbol{k})A_\alpha(q)A_{\alpha'}(q') \times \tilde{\Phi}(3,k,q+q'), \tag{20}$$

$$\mathcal{S}'_\theta = \frac{1}{\sqrt{\beta N_s}}\sum_{k,q} w(\boldsymbol{k},\boldsymbol{q})\theta(q)\tilde{\Phi}(2,k,q) + \frac{1}{\beta N_s}\sum_{k,q,q'} z(\boldsymbol{k},\boldsymbol{q},\boldsymbol{q}')\theta(q)\theta(q') \times \tilde{\Phi}(1,k,q+q'), \tag{21}$$

with

$$v_0(\boldsymbol{k}) = -e,\ v_\alpha(\boldsymbol{k}) = (-e)\frac{\partial\epsilon_{\boldsymbol{k}}}{\partial k_\alpha} \ \text{ for } \alpha = 1,2,3, \tag{22}$$

$$m^{-1}_{\alpha\alpha'}(\boldsymbol{k}) = \frac{e^2}{2}\frac{\partial^2\epsilon_{\boldsymbol{k}}}{\partial k_\alpha \partial k_{\alpha'}} \ \text{ for } \alpha,\alpha' = 1,2,3, \tag{23}$$

$$\epsilon_{\boldsymbol{k}} = -\sum_{\boldsymbol{r}_i - \boldsymbol{r}_j} t_{ij}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}, \tag{24}$$

$$w(\boldsymbol{k},\boldsymbol{q}) = \frac{1}{2}(\Delta_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}+\boldsymbol{q}}), \tag{25}$$

$$z(\boldsymbol{k},\boldsymbol{q},\boldsymbol{q}') = \frac{1}{8}(\Delta_{\boldsymbol{k}} + 2\Delta_{\boldsymbol{k}+\boldsymbol{q}} + \Delta_{\boldsymbol{k}+\boldsymbol{q}+\boldsymbol{q}'}), \tag{26}$$

$$\Delta_{\boldsymbol{k}} = \frac{\Delta_0}{2}\big[\cos(k_x) - \cos(k_y)\big], \tag{27}$$

$$\tilde{\Phi}(\alpha,k,q) = \bar{\Phi}(k+q)\boldsymbol{\sigma}_\alpha\Phi(k). \tag{28}$$

$\epsilon_{\boldsymbol{k}}$ is the hopping energy and $\Delta_{\boldsymbol{k}}$ the d-wave superconducting order parameter. $\boldsymbol{\sigma}_\alpha$ denotes for $\alpha = 1,2,3$ the Pauli matrices along the x, y, and z directions, respectively, and for $\alpha = 0$ the 2×2 identity matrix. n_α is equal to 3 for $\alpha = 0$ and zero otherwise. The summations in $\boldsymbol{k}$-space always extend over the first Brillouin zone.

The next step consists in performing the functional integration over the $\bar{\Psi}$, Ψ fermionic fields appearing in equation (18). It is achieved by noticing that $\mathcal{Z}[A,\theta]/\mathcal{Z}^{(0)}$ can be seen as a generating functional, θ, A representing external sources which couple to one-particle density operators $\tilde{\Phi}(\alpha,k,q)$. Abbreviating the set of variables $\{\alpha_1,k_1,q_1\}$, $\{\alpha_2,k_2,q_2\}$ symbolically by 1, 2, etc., we can write

$$\mathcal{S}'_A + \mathcal{S}'_\theta = \int d1 E(1)\tilde{\Phi}(1), \tag{29}$$

where an explicit expression for $E(1)$ can easily be read off from equations (20) and (21). The integration over Fermi fields yields then for $\mathcal{S}_{\mathit{eff}}$, defined in equation (13) [26],

$$\mathcal{S}_{\mathit{eff}}[A,\theta] = \int d1\ \mathcal{G}_c(1)E(1) - \frac{1}{2}\int d1d2\ \mathcal{G}_c(1;2)E(1)E(2) + ..., \tag{30}$$

$\mathcal{G}_c$ are connected Green's functions. They are related to the usual Green's functions $\mathcal{G}$, defined by

$$\mathcal{G}(1;...;n) = \int \mathcal{D}\bar{\Phi}\mathcal{D}\Phi\ \tilde{\Phi}(1)...\tilde{\Phi}(n)\mathrm{e}^{-\mathcal{S}_0}/\mathcal{Z}_0, \tag{31}$$

via the cumulant expansion of $\mathcal{G}_c$ in terms of $\mathcal{G}$ obtained from the identity [26]

$$\mathcal{W}[A,\theta] = \ln\Big(\frac{\mathcal{Z}[A,\theta]}{\mathcal{Z}^{(0)}}\Big), \tag{32}$$

where $\mathcal{W}[A,\theta]$ is the generating functional for connected Green's functions. Explicitly, one obtains $\mathcal{G}_c(1) = \mathcal{G}(1)$, $\mathcal{G}_c(1;2) = \mathcal{G}(1;2) - \mathcal{G}(1)\mathcal{G}(2)$, etc. We point out that in the general case, these connected Green's functions have

to be calculated in the presence of the V and U interaction terms.

Expressing E in terms of A and θ it is clear that no linear terms in A or θ can appear in $\mathcal{S}_{eff}$ for a non-vanishing momentum $\boldsymbol{q}$. The quadratic terms in A and θ of $\mathcal{S}_{eff}$ become

$$\mathcal{S}_{eff}[A,\theta] = \frac{1}{2}\sum_{q,\alpha,\alpha'} A_\alpha(q)\mathcal{K}^{AA}_{\alpha\alpha'}(q)A_{\alpha'}(-q) + \frac{1}{2}\sum_q \theta(q)\mathcal{K}^{\theta\theta}(q)\theta(-q) + \sum_{q,\alpha} A_\alpha(q)\mathcal{K}^{A\theta}_\alpha(q)\theta(-q), \tag{33}$$

with

$$\mathcal{K}^{AA}_{\alpha\alpha'}(q) = -\frac{1}{\beta N_s}\sum_{k,k'} v_\alpha(\boldsymbol{k})v_{\alpha'}(\boldsymbol{k}') \times\ \mathcal{G}(n_\alpha,k,q;n_{\alpha'},k',-q) + \frac{2}{\beta N_s}\sum_k m^{-1}_{\alpha\alpha'}(\boldsymbol{k})(1-\delta_{\alpha 0})(1-\delta_{\alpha' 0}) \times\ \mathcal{G}(3,k,0), \tag{34}$$

$$\mathcal{K}^{\theta\theta}(q) = -\frac{1}{\beta N_s}\sum_{k,k'} w(\boldsymbol{k},\boldsymbol{q})w(\boldsymbol{k}',-\boldsymbol{q}) \times\ \mathcal{G}(2,k,q;2,k',-q) + \frac{2}{\beta N_s}\sum_k z(\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q})\mathcal{G}(1,k,0), \tag{35}$$

$$\mathcal{K}^{A\theta}_\alpha(q) = -\frac{1}{\beta N_s}\sum_{k,k'} v_\alpha(\boldsymbol{k})w(\boldsymbol{k}',-\boldsymbol{q}) \times \mathcal{G}(n_\alpha,k,q;2,k',-q), \tag{36}$$

General considerations show that the Taylor expansion of $\mathcal{K}^{\theta\theta}$ in powers of q has no constant or linear terms, that of $\mathcal{K}^{A\theta}$ no constant terms. We therefore write

$$\mathcal{K}^{\theta\theta}(q) = -\sum_{\alpha,\alpha'=0}^{3} q_\alpha \mathcal{K}^{\theta\theta}_{\alpha\alpha'}(q)q_{\alpha'}, \tag{37}$$

$$\mathcal{K}^{A\theta}_\alpha(q) = -\sum_{\alpha'=0}^{3} \mathcal{K}^{A\theta}_{\alpha\alpha'}(q)q_{\alpha'}, \tag{38}$$

where the functions $\mathcal{K}^{\theta\theta}_{\alpha\alpha'}(q)$ and $\mathcal{K}^{A\theta}_{\alpha\alpha'}(q)$ approach in general finite values for $q \to 0$. Writing

$$\theta_\alpha(q) \equiv q_\alpha\theta(q), \tag{39}$$

for $\alpha = 0, ..., 3$, we obtain for S_{eff}

$$\mathcal{S}_{eff}[A,\theta] = \frac{1}{2}\sum_{q,\alpha,\alpha'}\Big(A_\alpha(q)\mathcal{K}^{AA}_{\alpha\alpha'}(q)A_{\alpha'}(-q) + \theta_\alpha(q)\mathcal{K}^{\theta\theta}_{\alpha\alpha'}(q)\theta_{\alpha'}(-q) + 2A_\alpha(q)\mathcal{K}^{A\theta}_{\alpha\alpha'}(q)\theta_{\alpha'}(-q)\Big). \tag{40}$$

Equation (40) shows that $\mathcal{S}_{eff}$ actually depends not on the phase θ itself but only on its gradients. The three functions $\mathcal{K}^{AA}$, $\mathcal{K}^{\theta\theta}$, and $\mathcal{K}^{A\theta}$ are not independent from each other. To see this we apply a gauge transformation to $\mathcal{S}_{eff}$ [19]

$$A_\alpha(q) \to A_\alpha(q) + \mathrm{i}\chi_\alpha(q), \quad \theta_\alpha(q) \to \theta_\alpha(q) - 2e\chi_\alpha(q), \tag{41}$$

where both equations hold for $\alpha = 0, 1, 2, 3$ and we have defined $\chi_\alpha(q) \equiv q_\alpha\chi(q)$. For this gauge transformation it is sufficient to consider both θ and χ to be very small. As a result we only deal with small phase deviations from $\theta = 0$ and a possible non-uniqueness of wave functions cannot play any role. Invariance of $\mathcal{S}_{eff}$ against this transformation yields the identities

$$\mathcal{K}^{AA}_{\alpha\alpha'}(q) = -4e^2\mathcal{K}^{\theta\theta}_{\alpha\alpha'}(q) = -2\mathrm{i}e\mathcal{K}^{A\theta}_{\alpha\alpha'}(q). \tag{42}$$

An explicit calculation for the case of non-interacting electrons is presented in the appendix and provide direct checks of equation (42). Using the above relations one finally finds for $\mathcal{S}_{eff}$

$$\mathcal{S}_{eff}[A,\theta] = \frac{1}{2}\sum_{q,\alpha,\alpha'}\left[A_\alpha(q) + \frac{\mathrm{i}}{2e}\theta_\alpha(q)\right]\mathcal{K}^{AA}_{\alpha\alpha'}(q) \times \left[A_{\alpha'}(-q) + \frac{\mathrm{i}}{2e}\theta_{\alpha'}(-q)\right]. \tag{43}$$

The matrix $\mathcal{K}^{AA}_{\alpha\alpha'}$ connects the charge and current induced by an applied external potential A. Our microscopic expression, equation (34), just represents a generalization of the usual expression [38] to the case with interactions between electrons. Some contributions to the kernel $\mathcal{K}$, for instance, the second term on the right-hand side of equation (34) develop non-analytic behavior at small $\boldsymbol{q}$'s such as square root singularities due to the linear fermionic dispersion near the nodal direction [39]. It also has been pointed out that this dispersion may lead to instabilities and new phases such as spin density or Cooper pair charge density wave states [18,40]. $\mathcal{K}^{AA}_{00}(q)$ is the density-density correlation function. For a charged system the Coulomb interaction in equation (2) is important. The diagrams to $\mathcal{K}^{AA}_{00}(q)$ can be classified to be reducible or irreducible depending whether or not the diagram decomposes into two unconnected parts by cutting on Coulomb line [45]. Denoting by $\tilde{\mathcal{K}}$ its irreducible part the exact density-density Green's function $\mathcal{K}^{AA}_{00}(q)$ can be obtained by summing a geometrical series, i.e., by,

$$\mathcal{K}^{AA}_{00}(q) = \frac{\tilde{\mathcal{K}}^{AA}_{00}(q)}{1 + V(\boldsymbol{q})\tilde{\mathcal{K}}^{AA}_{00}(q)}. \tag{44}$$

Taking into account the layered structure of high-T_c cuprates, the Coulomb potential $V(\boldsymbol{q})$ is given by [46]

$$V(\boldsymbol{q}) = \frac{2\pi e^2 d}{\epsilon_b q_\parallel}\Big[\frac{\sinh(q_\parallel d)}{\cosh(q_\parallel d) - \cos(q_z d)}\Big], \tag{45}$$

with $q_\parallel = \sqrt{q_x^2 + q_y^2}$. d is the distance between layers and ϵ_b a background dielectric constant. $\mathcal{K}^{AA}_{00}$ approaches at

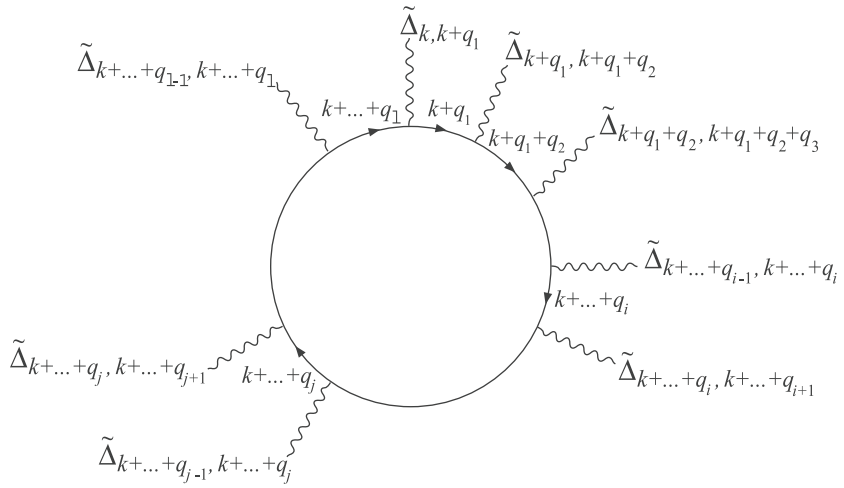

Fig. 1. Ring diagram of order ℓ in the order parameter $\tilde{\Delta}_{k,k+q}$. The solid lines denote the unperturbed Green's function $\mathcal{G}_0$.

small wave vectors the universal function $1/V(\boldsymbol{q})$. Symmetry requires $\mathcal{K}^{AA}_{0\alpha'} = 0$ for $\alpha' = 1,2,3$. Furthermore, $\mathcal{K}^{AA}_{\alpha\alpha'}$ is for $\alpha,\alpha' = 1,2,3$ already irreducible in the above sense so that no analogue to equation (44) exists in this case.

Finally we need an equation to determine Δ_0. Using the Nambu formulation the terms in the small square bracket in equation (6) can be written after a Fourier transformation as

$$-\frac{1}{4}\int_0^\beta d\tau \sum_{\langle ij\rangle}[\Delta_{ij}\bar{B}^*_{ij}(\tau) + \Delta^*_{ij}(\tau)\bar{B}_{ij}(\tau)] = \sum_{k,q}\bar{\Phi}_k\tilde{\Delta}_{k,k+q}\Phi_{k+q}. \quad (46)$$

$\tilde{\Delta}_{k,k+q}$ is, except for a factor $-1/4$, the Fourier transform of $\Delta_{ij}(\tau)\boldsymbol{\sigma}^+ + \Delta^*_{ij}(\tau)\boldsymbol{\sigma}^-$ with $\boldsymbol{\sigma}^\pm = (\boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2)/2$. In determining Δ_0 we may put all phases to zero so that $\Delta_{ij}(\tau)\boldsymbol{\sigma}^+ + \Delta^*_{ij}(\tau)\boldsymbol{\sigma}^-$ reduces to $\Delta_0\gamma_{ij}\boldsymbol{\sigma}_1$. Considering the deviation $\delta\Delta_0$ from the saddle point value Δ_0 the corresponding change in the action on the right-hand side of equation (46) yields a change linear in $\delta\Delta_0$ in $\mathcal{S}_{eff}$ after integrating out the Fermi fields as previously. Requiring that the total linear term in $\delta\Delta_0$ vanishes yields the desired equation

$$\frac{\Delta_0}{J} = \frac{1}{\beta N_s}\sum_k(\cos k_x - \cos k_y)\,\mathcal{G}(1,k,0). \quad (47)$$

4 Gradient expansion of $\mathcal{S}_{\mathrm{eff}}$

$\mathcal{S}_{eff}$ has been derived in Section 3 under the assumption that the phase θ is small. An expansion of $\mathcal{S}_{eff}$ in terms of gradients of the order parameter seems to be more satisfying because θ is then no longer restricted to small values. As shown in reference [41] for the case of a charge-density-wave state, resummations allow to transform the expansion of $\mathcal{S}_{eff}$ in powers of the order parameter into an expansion in powers of gradients of the order parameter. In the following we adapt this method to our action and consider first the non-interacting case $\mathcal{S}_{int} = 0$. The integration over fermions can then easily be performed and one obtains, dropping the constant $\sim\Delta_0^2$,

$$\begin{aligned}\mathcal{S}_{eff}[0,\theta] &= -\mathrm{Tr}\Big\{\ln(-\mathcal{G}_0^{-1} + \tilde{\Delta})\Big\} \quad (48)\\ &= -\mathrm{Tr}\Big\{\ln(-\mathcal{G}_0^{-1})\Big\} + \sum_{\ell=1}^{\infty}\frac{1}{\ell}\mathrm{Tr}\Big\{(\mathcal{G}_0\tilde{\Delta})^\ell\Big\}.\end{aligned}$$

$\mathcal{G}_0$ is the unperturbed Green's function due to the two first terms in equation (15), its expression is given in equation (A.1). Tr denotes the trace over k and the Nambu index. The ℓth order term X_ℓ in the sum over ℓ can be written in frequency-momentum space as

$$X_\ell = \frac{1}{\ell}\sum_{q_1,\ldots,q_\ell}\int dr e^{\mathrm{i}r\cdot(q_1+..+q_\ell)}Y(q_1,q_2,...,q_\ell), \quad (49)$$

$$\begin{aligned}Y(q_1,q_2,...,q_\ell) = \sum_k \mathrm{Tr}'\Big\{&\tilde{\Delta}_{k,k+q_1}\mathcal{G}_0(k+q_1)\ ...\\ &\times\tilde{\Delta}_{k+...q_{\ell-1},k+...+q_\ell}\\ &\times\mathcal{G}_0(k+...+q_\ell)\Big\}. \quad (50)\end{aligned}$$

r stands for the vector $(\tau,\boldsymbol{r}_i)$, and $\int dr$ for $1/(\beta N_s)\sum_i\int_0^\beta d\tau$. Tr$'$ denotes a trace over the Nambu index. X_ℓ can be visualized by a ring diagram (see Fig. 1) where the electronic Green's function $\mathcal{G}_0$ (solid line) is scattered at the external potentials $\tilde{\Delta}$ (wavy lines).

For the following it is convenient to write $\tilde{\Delta}(k,q_1)$ instead of $\tilde{\Delta}_{k,k+q_1}$. The first momentum k refers then to the relative and the second one q_1 to the center-of-mass motion of the Cooper pair. k may assume arbitrary values whereas $q_1,q_2,...,$ are considered to be small.

In the non-interacting case the calculations for $\mathcal{S}_{eff}$ in Section 3 correspond to the evaluation of ring diagrams of the type shown in Figure 1. To obtain $\mathcal{S}_{eff}$ in second order in θ we had to take into account one and also

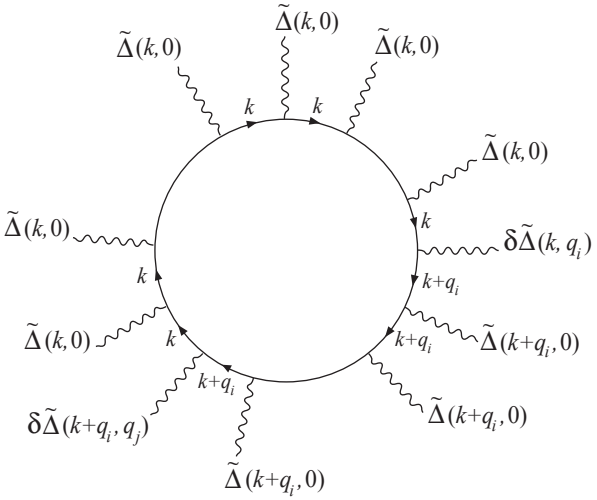

Fig. 2. Ring diagram of order ℓ in the order parameter $\tilde{\Delta}_{k,k+q}$ with momenta corresponding to small phase fluctuations around an equilibrium state with long-range order.

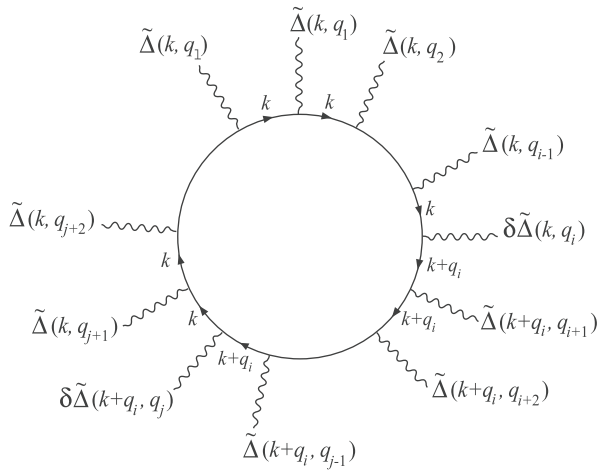

Fig. 3. Ring diagram of order ℓ in the order parameter $\tilde{\Delta}_{k,k+q}$ with momenta corresponding to the leading non-local contribution in the gradient expansion.

two non-equilibrium external lines due to phase fluctuations yielding the second and first terms in equation (35), respectively. The corresponding ring diagram of order ℓ is obtained by evaluating the Green's functions in equations (34)–(36) for non-interacting electrons and expanding them to order $\ell-2$ or $\ell-1$, respectively, in $\tilde{\Delta}(k,0)$. Let us first consider the first term in more detail. Its ℓth order contribution can be illustrated by the diagram shown in Figure 2. $\delta\tilde{\Delta}(k,q_i)$ and $\delta\tilde{\Delta}(k+q_i,q_j)$ are the two non-equilibrium lines with $q_i=-q_j$ from momentum conservation. Between these lines the electrons are $\ell-2$ times scattered at the equilibrium order parameter with zero momentum. In this case the phase deviations from their equilibrium values must be considered as small so that the expansion in powers of θ is appropriate. If the phase does not exhibit true long-range order but still varies slowly in space and time the ring diagrams of Figure 1 have to be evaluated in a different way. In this case the momenta q_i in the external lines have to be kept but one may expand the electron propagators in powers of q_i. As shown in references [41,42] such an expansion generates a gradient expansion for $\mathcal{S}_{eff}$. In particular, it does not assume that θ is small but only that the gradients of θ are small. Figure 3 shows the distribution of momenta which yield the leading term in this expansion. Scattering at the sites i and j is accompanied by a change in energy and momentum q_i, whereas no change in energy or momentum occurs at all the other sites. Keeping higher terms in the expansion of the electron propagator in terms of q_i would yield contributions to $\mathcal{S}_{eff}$ which are at least of third order in phase gradients.

Evaluating the diagram of Figure 3 the sums over $q_1,...,q_{i-1},q_{i+1},...,q_{j-1},q_{j+1},...q_\ell$ can be immediately be carried out yielding products of $\tilde{\Delta}(k,r)$, the Fourier transform of $\tilde{\Delta}(k,k_1)$ with respect to k_1. It is convenient to introduce a Green's function $\tilde{\mathcal{G}}(k,r)$ by

$$\tilde{\mathcal{G}}^{-1}(k,r)=\mathcal{G}_0^{-1}(k)-\tilde{\Delta}(k,r). \tag{51}$$

After a small rearrangement of terms one obtains for the diagram the expression

$$X_\ell^{(2)}=\frac{1}{2\ell}\sum_{i\neq j}\sum_{k,q_i}\int dr\ \mathrm{Tr}'\Big\{\tilde{\mathcal{G}}^{(\ell-j+i-1)}(k,r)\tilde{\Delta}(k,q_i)\times\tilde{\mathcal{G}}^{(j-i-1)}(k+q_i,r)\times\tilde{\Delta}(k,-q_i)\Big\}, \tag{52}$$

where $\tilde{\mathcal{G}}^{(\ell)}(k,r)$ is the contribution of order ℓ to $\tilde{\mathcal{G}}$. The sums over i,j and finally also over ℓ can also be performed yielding the following contribution to $\mathcal{S}_{eff}$

$$\mathcal{S}_{eff}^{(2)}=\frac{1}{2}\sum_{k,q}\int dr\mathrm{Tr}'\Big\{\tilde{\mathcal{G}}(k,r)\tilde{\Delta}(k,q)\times\tilde{\mathcal{G}}(k+q,r)\tilde{\Delta}(k,-q)\Big\}. \tag{53}$$

Ring diagrams with only one non-equilibrium line $\delta\tilde{\Delta}(k,q)$ can be evaluated in a similar manner. Though these diagrams only involve the $q=0$ Fourier component of $\tilde{\Delta}$ due to energy and momentum conservation they yield products of phase fluctuations when passing from order parameter to phase fluctuations. Evaluating these ring diagrams in form of a gradient expansion yields the following contribution to $\mathcal{S}_{eff}$

$$\mathcal{S}_{eff}^{(1)}=\sum_k\int dr\ \mathrm{Tr}'\Big\{\tilde{\mathcal{G}}(k,r)\tilde{\Delta}(k,0)\Big\}. \tag{54}$$

Equations (53) and (54) represent time and space averages of the corresponding homogenous action where the Green's functions contain the local instead of the global gap. This gives a simple recipe to generalize an expression for $\mathcal{S}_{eff}$ derived under the assumption of small phase fluctuations to one which is valid also for large phase fluctuations in the leading order of a gradient expansion: one writes $\mathcal{S}_{eff}$ as a density in space and time and then uses

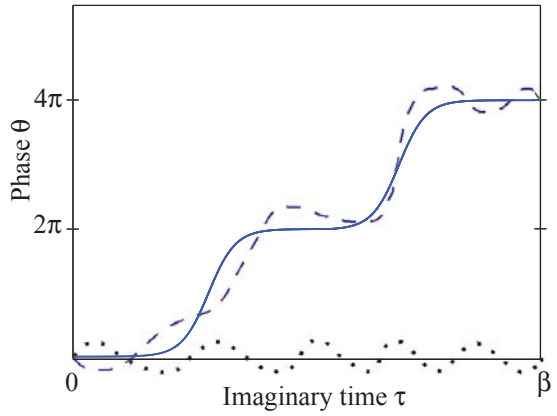

Fig. 4. (Color online) Small and large fluctuations in the θ-τ plane.

at a given point r in space and time the homogenous expression for $\mathcal{S}_{\mathit{eff}}$ with the constant gap value $\tilde{\Delta}(k,r)$ in the Green's functions. In the phase-only approximation order parameter fluctuations are solely due to fluctuations in the phase. Because the Green's functions in equation (53) refer for a given r to a constant phase this phase can be gauged away by a global gauge transformation without any additional contribution to the vector potential A. The right-hand side of equation (53) is clearly invariant against a global gauge transformation. This is true even separately for the product of the order parameters and the susceptibility because the latter involves the same number of creation and annihilation operators so that global phases cancel. In the phase-only approximation we thus may put the phase in equation (53) to zero. This means that the ring diagram of Figure 4 reduces to a diagram of second-order in $\tilde{\Delta}$ where the Green's functions (solid lines) are to be calculated with the equilibrium order parameter equations (7) and (8). This result proves that within the phase-only approximation the effective action derived for small fluctuations in Section 3 is in leading order in field gradients identical with that of the gradient expansion. In the general case where also the amplitude Δ_0 varies in time and space equation (53) clearly differs from the corresponding lowest-order expression in Section 3.

It is evident that the above results also hold for interacting electrons. For a given skeleton diagram in interaction and electron lines there is again a one-to-one correspondence between diagrams in ℓth order in $\tilde{\Delta}$ for ordered and disordered ground states. For ordered ground states this diagram is obtained by expanding electron lines in $\tilde{\Delta}(k,0)$ so that the total order is $\ell-2$. In the disordered case the same diagram is obtained by expanding $\mathcal{S}_{\mathit{eff}}$ in equation (30) up to the order ℓ, and associating in all possible ways two vertices with and the remaining $\ell-2$ without energy and momentum changes. Writing the energy and momentum conservation as a Fourier integral over r, the momentum integration at the $\ell-2$ vertices without energy and momentum changes can be carried out yielding products of $\tilde{\Delta}(k,r)$. As a result a strict one-to-one correspondence between diagrams of the ordered and disordered cases is established from which the above recipe follows.

5 Discussion and conclusions

Equation (43) together with equations (34)–(38) represent a microscopic expression for the effective action of phase fluctuations in a d-wave superconductor with interacting electrons. In Section 3 this expression was derived under the assumption that θ is small. The dotted line in Figure 4 illustrates this case for the τ-dependence of θ. Within the interval $[0,\beta]$ θ performs small oscillations around the equilibrium value $\theta=0$. Similar pictures can be drawn for paths in $r=(\tau,\boldsymbol{r})$ space. The gradient expansion in Section 4 allows to consider more general paths of the form $\theta(r)=\theta_0(r)+\delta\theta(r)$. $\theta_0(r)$ is assumed to be slowly varying with r whereas $\delta\theta(r)$ must be small, as illustrated by the solid and dashed lines, respectively, in Figure 4. Since in the phase-only approximation the gradient expansion leads to the same expression for $\mathcal{S}_{\mathit{eff}}$ as the perturbation theory equation (43) is actually valid for all paths shown in Figure 4. To determine $\mathcal{S}_{\mathit{eff}}$ for the solid curve it is sufficient to take in equation (43) the long-wavelength, low-frequency limit in $K^{AA}_{\alpha\alpha'}(q)$. The important slowly varying extremal paths satisfy then a second-order differential equation which may have besides of trivial constant also vortex solutions illustrated by the solid line in Figure 4.

$\mathcal{S}_{\mathit{eff}}$ reduces in the static limit to the Ginzburg-Landau form for the phase-dependent part of the free energy of a superconductor. For non-interacting electrons it also coincides with previous expressions derived using singular gauge transformations. Note that our derivation of $\mathcal{S}_{\mathit{eff}}$ avoided any singular gauge transformation. Instead we expanded both the diagonal and the off-diagonal parts of the microscopic action in terms of gradients in the phase and related the two parts using non-singular gauge transformations. The fact that our final $\mathcal{S}_{\mathit{eff}}$ agrees for non-interacting electrons with previously derived effective actions implies that the change of boundary conditions associated with singular gauge transformations does not enter $\mathcal{S}_{\mathit{eff}}$ as long as the field gradients are small and only second-order terms of them are retained. Heuristically, the above expression for $\mathcal{S}_{\mathit{eff}}$ can be obtained in the following simple way: Discard order parameter fluctuations in the original microscopic Lagrangian but keep the potential A. Integrate then over the Fermi fields and make the resulting $\mathcal{S}_{\mathit{eff}}$ gauge-invariant by applying the gauge transformation, equation (41), and interpret the field $-2e\chi$ as the phase field of the order parameter.

We are grateful to Dirk Manske for a careful reading of the manuscript.

Appendix A: Interrelation between different kernels $\mathcal{K}$ in $\mathcal{S}_{\mathsf{eff}}$ for non-interacting electrons

In this appendix we check the relation between the kernels $\mathcal{K}^{AA}$ and $\mathcal{K}^{\theta\theta}$ given by equation (42) for an isolated layer

and non-interacting electrons. The free Green's function matrix is then given by

$$\mathcal{G}_0(k) = \frac{-1}{(\nu_m)^2 + (\xi_{\boldsymbol{k}})^2 + (\Delta_{\boldsymbol{k}})^2} \times \begin{bmatrix} \mathrm{i}\nu_m + \xi_{\boldsymbol{k}} & -\Delta_{\boldsymbol{k}} \\ -\Delta_{\boldsymbol{k}} & \mathrm{i}\nu_m - \xi_{\boldsymbol{k}} \end{bmatrix}, \quad \text{(A.1)}$$

with the following energies

$$\Delta_{\boldsymbol{k}} = \frac{\Delta_0}{2}\big[\cos(k_x) - \cos(k_y)\big],\ \xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu, \quad \text{(A.2)}$$
$$\epsilon_{\boldsymbol{k}} = -2t\big[\cos(k_x) + \cos(k_y)\big] - 4t^{'}\big[\cos(k_x).\cos(k_y)\big],$$

where $\epsilon_{\boldsymbol{k}}$ includes nearest and next-nearest neighbor hopping contributions. Equation (47) yields the BCS gap equation [43,38]

$$\frac{1}{J} = \frac{1}{N_s}\sum_{\boldsymbol{k}} \frac{\big[\cos(k_x) - \cos(k_y)\big]^2}{2E_{\boldsymbol{k}}} \tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right), \quad \text{(A.3)}$$

where $E_{\boldsymbol{k}}$ is the quasiparticle energy, $E_{\boldsymbol{k}} = \sqrt{(\xi_{\boldsymbol{k}})^2 + (\Delta_{\boldsymbol{k}})^2}$, and the summation in $\boldsymbol{k}$-space is extended over the first (square) Brillouin zone. Performing the summation over the fermionic Matsubara frequencies we find for the kernel $\mathcal{K}^{\theta\theta}$ of equation (35),

$$\begin{aligned}\mathcal{K}^{\theta\theta}(q) = \frac{1}{4\mathrm{N_s}}\sum_{\boldsymbol{k}} \Bigg(& \big\{\Delta_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}-\boldsymbol{q}}\big\}^2 \\ & \times \Bigg[\frac{\big\{1 - n_{\mathrm{FD}}(E_{\boldsymbol{k}-\boldsymbol{q}}) - n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\}}{2} \\ & \times \left\{1 + \frac{\xi_{\boldsymbol{k}}\xi_{\boldsymbol{k}-\boldsymbol{q}} + \Delta_{\boldsymbol{k}}\Delta_{\boldsymbol{k}-\boldsymbol{q}}}{E_{\boldsymbol{k}}E_{\boldsymbol{k}-\boldsymbol{q}}}\right\} \\ & \times \left\{\frac{1}{\mathrm{i}\omega_n - E_{\boldsymbol{k}} - E_{\boldsymbol{k}-\boldsymbol{q}}} - \frac{1}{\mathrm{i}\omega_n + E_{\boldsymbol{k}} + E_{\boldsymbol{k}-\boldsymbol{q}}}\right\} \\ & + \frac{\big\{n_{\mathrm{FD}}(E_{\boldsymbol{k}-\boldsymbol{q}}) - n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\}}{2} \\ & \times \left\{1 - \frac{\xi_{\boldsymbol{k}}\xi_{\boldsymbol{k}-\boldsymbol{q}} + \Delta_{\boldsymbol{k}}\Delta_{\boldsymbol{k}-\boldsymbol{q}}}{E_{\boldsymbol{k}}E_{\boldsymbol{k}-\boldsymbol{q}}}\right\} \\ & \times \left\{\frac{1}{\mathrm{i}\omega_n - E_{\boldsymbol{k}} + E_{\boldsymbol{k}-\boldsymbol{q}}} - \frac{1}{\mathrm{i}\omega_n + E_{\boldsymbol{k}} - E_{\boldsymbol{k}-\boldsymbol{q}}}\right\}\Bigg] \\ & + \frac{2\Delta_{\boldsymbol{k}}\big\{\Delta_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}-\boldsymbol{q}}\big\}}{E_{\boldsymbol{k}}}\big\{1 - 2n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\}\Bigg), \end{aligned} \quad \text{(A.4)}$$

with n_{FD} being the Fermi-Dirac distribution function.

To perform a first explicit check of equation (42), we consider the frequency dependence of the kernels $\mathcal{K}^{\theta\theta}$, $\mathcal{K}^{AA}$ and $\mathcal{K}^{A\theta}$. With equations (37) and (A.4) we have

$$\mathcal{K}^{\theta\theta}_{00}(\mathrm{i}\omega_n, \mathbf{0}) = -\frac{1}{\mathrm{N_s}}\sum_{\boldsymbol{k}}\big\{1 - 2n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\} \times \frac{(\Delta_{\boldsymbol{k}})^2}{E_{\boldsymbol{k}}\big\{(\mathrm{i}\omega_n)^2 - 4(E_{\boldsymbol{k}})^2\big\}}. \quad \text{(A.5)}$$

Using equation (34) we get for the electromagnetic field kernel

$$\mathcal{K}^{AA}_{00}(\mathrm{i}\omega_n, \mathbf{0}) = \frac{e^2}{\beta N_s}\sum_{k}\mathrm{Tr}'\Big[\boldsymbol{\sigma}_3\mathcal{G}_0(k)\boldsymbol{\sigma}_3\mathcal{G}_0(k)\Big], \quad \text{(A.6)}$$

which gives after summation over the fermionic frequencies

$$\mathcal{K}^{AA}_{00}(\mathrm{i}\omega_n, \mathbf{0}) = \frac{4e^2}{N_s}\sum_{\boldsymbol{k}}\big\{1 - 2n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\} \times \frac{(\Delta_{\boldsymbol{k}})^2}{E_{\boldsymbol{k}}\big\{(\mathrm{i}\omega_n)^2 - 4(E_{\boldsymbol{k}})^2\big\}}. \quad \text{(A.7)}$$

Similarly, equation (36) yields,

$$\mathcal{K}^{A\theta}_{0}(q) = \frac{e}{2\beta\mathrm{N_s}}\sum_{k}(\Delta_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}+\boldsymbol{q}}) \times \mathrm{Tr}'\Big[\boldsymbol{\sigma}_3\mathcal{G}_0(k)\boldsymbol{\sigma}_2\mathcal{G}_0(k+q)\Big]. \quad \text{(A.8)}$$

Performing the summation over the fermionic frequencies and putting $\boldsymbol{q} = \mathbf{0}$ we find,

$$\mathcal{K}^{A\theta}_{00}(\mathrm{i}\omega_n, \mathbf{0}) = \frac{2\mathrm{i}}{N_s}\sum_{\boldsymbol{k}}\big\{1 - 2n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\} \times \frac{(\Delta_{\boldsymbol{k}})^2}{E_{\boldsymbol{k}}\big\{(\mathrm{i}\omega_n)^2 - 4(E_{\boldsymbol{k}})^2\big\}}. \quad \text{(A.9)}$$

By comparing equations (A.5), (A.7) and (A.9) one can see immediately

$$\begin{aligned}\mathcal{K}^{AA}_{00}(\mathrm{i}\omega_n, \mathbf{0}) &= -4e^2\mathcal{K}^{\theta\theta}_{00}(\mathrm{i}\omega_n, \mathbf{0}) \\ &= -2\mathrm{i}e\mathcal{K}^{A\theta}_{00}(\mathrm{i}\omega_n, \mathbf{0}), \end{aligned} \quad \text{(A.10)}$$

in agreement with equation (42), for any frequency $\mathrm{i}\omega_n$.

The low-frequency limit of the effective action $\mathcal{S}_{\mathit{eff}}[0,\theta]$ can be investigated by expanding quadratically its kernel. We start by considering the zero-temperature case. The low-frequency expansion up to the second-order in $\mathrm{i}\omega_n$ of $\mathcal{K}^{\theta\theta}$, equation (A.4), is straightforward, we have

$$\mathcal{K}^{\theta\theta}(\mathrm{i}\omega_n \to 0, \boldsymbol{q} = \mathbf{0}) = -\frac{1}{4N_s}\sum_{\boldsymbol{k}}\frac{(\Delta_{\boldsymbol{k}})^2}{(E_{\boldsymbol{k}})^3}(\mathrm{i}\omega_n)^2. \quad \text{(A.11)}$$

A similar procedure can be repeated in the finite temperature case. The low-frequency limit of $\mathcal{K}^{\theta\theta}$ is given by

$$\mathcal{K}^{\theta\theta}(\mathrm{i}\omega_n \to 0, \boldsymbol{q} = \mathbf{0}) = \frac{1}{4}\chi_0(\mathrm{i}\omega_n \to 0, \boldsymbol{q} = \mathbf{0}) \cdot (\mathrm{i}\omega_n)^2, \quad \text{(A.12)}$$

with χ_0 the mean field density-density correlation function

$$\chi_0(\mathrm{i}\omega_n \to 0, \boldsymbol{q} = \mathbf{0}) = -\frac{1}{N_s}\sum_{\boldsymbol{k}}\big\{1 - 2n_{\mathrm{FD}}(E_{\boldsymbol{k}})\big\}\frac{(\Delta_{\boldsymbol{k}})^2}{(E_{\boldsymbol{k}})^3}. \quad \text{(A.13)}$$

Our expressions (A.11) and (A.13) can be compared with equations (16) and (19) of reference [14], taken in the low-frequency limit which we considered. We obtain the same results. Therefore we can conclude that $\mathcal{K}^{AA}$ and $\mathcal{K}^{\theta\theta}$ are equal, up to a constant factor proportional to the electronic charge. This provides an additional non-trivial check of equation (42).

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